# Nilmanifolds with a calibrated $\mathrm{G}_2$-structure

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**Abstract**

We introduce obstructions to the existence of a calibrated $\mathrm{G}_2$-structure on a Lie algebra $\mathfrak{g}$ of dimension seven, not necessarily nilpotent. In particular, we prove that if there is a Lie algebra epimorphism from $\mathfrak{g}$ to a six-dimensional Lie algebra $\mathfrak{h}$ with kernel contained in the center of $\mathfrak{g}$, then $\mathfrak{h}$ has a symplectic form. As a consequence, we obtain a classification of the nilpotent Lie algebras that admit a calibrated $\mathrm{G}_2$-structure.

**MSC classification**: Primary 53C38; Secondary 53C15, 17B30

**Key words**: calibrated $G_2$ forms, nilpotent Lie algebras, Lefschetz property

## 1 Introduction

A Riemannian 7-manifold with holonomy contained in $\mathrm{G}_2$ can be characterized by the existence of an associated parallel 3-form. The first examples of complete metrics with holonomy $\mathrm{G}_2$ were given by Bryant and Salamon in [4], and the first examples of compact manifolds with such a metric were constructed by Joyce in [16]. Explicit examples on solvable Lie groups were constructed in [5]; examples on nilpotent Lie groups can be obtained by taking a nilpotent six-dimensional group with a half-flat structure and solving Hitchin's evolution equations (see [9, 15]). More generally, one can consider $\mathrm{G}_2$-structures where the associated 3-form $\varphi$ is closed: then $\varphi$ defines a calibration ([14]), and the $\mathrm{G}_2$-structure is said to be *calibrated*. An equivalent condition is that the intrinsic torsion lies in the component $\mathcal{X}_2 \cong \mathfrak{g}_2$ ([10]).

Compact calibrated $\mathrm{G}_2$ manifolds have interesting curvature properties. It is well known that a $\mathrm{G}_2$ holonomy manifold is Ricci-flat, or equivalently, both Einstein and scalar-flat. On a compact calibrated $\mathrm{G}_2$ manifold, both the Einstein condition ([6]) and scalar-flatness ([3]) are equivalent to the holonomy being contained in $\mathrm{G}_2$. In fact, [3] shows that the scalar curvature is always non-positive.

Constructing examples is not a straightforward task. For instance, [7] classifies calibrated $\mathrm{G}_2$-manifolds on which a simple groups acts with cohomogeneity one, and no compact manifold occurs in this list. On the other hand, the second author exhibited the first example of a compact calibrated $\mathrm{G}_2$-manifold that

does not have holonomy $G_2$ [11]. This example is given in terms of a nilpotent Lie algebra $\mathfrak{g}$ and an element of $\Lambda^3\mathfrak{g}^*$ that corresponds to a closed left-invariant 3-form on the associated simply-connected Lie group. Since the structure constants are rational, there exists a uniform discrete subgroup [18]; the quotient, called a nilmanifold, has an induced calibrated $G_2$-structure.

In this paper we pursue this approach, and classify the nilpotent 7-dimensional Lie algebras that admit a calibrated $G_2$-structure. Since the structure constants turn out to be rational, each Lie algebra determines a nilmanifold. So, we obtain 12 compact calibrated $G_2$ nilmanifolds (see Theorem 4, Lemma 5 and Lemma 6). Three of them are reducible: they are the product of a circle with a symplectic half-flat nilmanifold, the latter being classified in [8]. The remaining nine are new.

The proof is based on two necessary conditions that a Lie algebra must satisfy for a calibrated $G_2$-structure to exist (see Proposition 1 and Lemma 3).

Our first obstruction is related to a construction of [1]. Suppose that $M$ is a 7-manifold with a calibrated $G_2$ form $\varphi$, and $X$ is a unit Killing field, i.e. $\mathcal{L}_X\varphi = 0$. Then if $\eta = X^\flat$, we can write

$$\varphi = \omega \wedge \eta + \psi^+,$$

where $\omega$, $\psi+$ and $d\eta$ are basic forms with respect to the 1-dimensional foliation defined by $X$. Suppose in addition that $X$ is the fundamental vector field of a free $S^1$ action; then basic forms can be identified with forms on $M/S^1$. By

$$0 = \mathcal{L}_X\varphi = d(X \lrcorner\, \varphi) = d\omega,$$

$\omega$ is a symplectic form on $M/S^1$. Moreover

$$0 = \omega \wedge d\eta + d\psi^+$$

implies that $[d\eta]$ is in the kernel of the map

$$H^2(M/S^1) \to H^4(M/S^1), \quad [\beta] \to [\beta \wedge \omega].$$

If this map is an isomorphism, then the $S^1$-bundle is trivial: this puts topological restrictions on $M$, which translate to algebraic conditions in our setup. A similar method was used in [8].

In principle, these restrictions reduce our problem to the classification of symplectic nilpotent Lie algebras of dimension six for which the map $H^2 \to H^4$ is non-injective (see the remark before Lemma 3). The complexity of the required calculations, however, motivate a different approach. In analogy with [9], we introduce a second obstruction, that requires computing the space of closed 3-forms. It consists in the observation that $(X \lrcorner\, \varphi)^3$ must be nonzero, whenever $X$ is a nonzero vector and $\varphi$ a 3-form defining a $G_2$-structure.

The final ingredient is Gong's classification of 7-dimensional indecomposable nilpotent Lie algebras ([12]). This list contains 140 Lie algebras and 9 one-parameter families; in addition, there are 35 decomposable nilpotent Lie algebras ([17], [19]). Calculations on a case-by-case basis show that our list of 12 examples is complete.

## 2 Calibrated $\mathrm{G}_2$-structures and obstructions

In this section we show obstructions to the existence of a calibrated $\mathrm{G}_2$ form on a Lie algebra (not necessarily nilpotent). First, we recall some definitions and results about $G_2$-structures.

Let us consider the space $\mathbb{O}$ of the Cayley numbers, which is a non-associative algebra over $\mathbb{R}$ of dimension 8. Thus, we can identify $\mathbb{R}^7$ with the subspace of $\mathbb{O}$ consisting of pure imaginary Cayley numbers. Then, the product on $\mathbb{O}$ defines on $\mathbb{R}^7$ the 3-form given by

$$e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245} \tag{1}$$

(see [10] and [13] for details), where $\{e^1, \ldots, e^7\}$ is the standard basis of $(\mathbb{R}^7)^*$. Here, $e^{127}$ stands for $e^1 \wedge e^2 \wedge e^7$, and so on. The group $G_2$ is the stabilizer of (1) under the standard action of $\mathrm{GL}(7, \mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. $G_2$ is one of the exceptional Lie groups, and it is a compact, connected, simply connected simple Lie subgroup of $SO(7)$ of dimension 14.

A $\mathrm{G}_2$-structure on a 7-dimensional Riemannian manifold $(M, g)$ is a reduction of the structure group $O(7)$ of the frame bundle to $G_2$. Manifolds admitting a $\mathrm{G}_2$-structure are called $\mathrm{G}_2$ manifolds. The existence of a $\mathrm{G}_2$-structure on $(M, g)$ is determined by a global 3-form $\varphi$ (the $G_2$ form) which can be locally written as (1) with respect to some (local) basis $\{e^1, \ldots, e^7\}$ of the (local) 1-forms on $M$. We say that the $\mathrm{G}_2$ manifold $M$ has a *calibrated $G_2$-structure* if there is a $G_2$-structure on $M$ such that the 3-form $\varphi$ is closed, and so $\varphi$ defines a calibration [14].

If $G$ is a 7-dimensional Lie group with Lie algebra $\mathfrak{g}$, then a $\mathrm{G}_2$-structure on $G$ is left-invariant if and only if the corresponding 3-form is left-invariant. Thus, a left invariant $\mathrm{G}_2$-structure on $G$ corresponds to an element $\varphi$ of $\Lambda^3\mathfrak{g}^*$ that can be written as (1) with respect to some coframe $\{e^1, \ldots, e^7\}$ on $\mathfrak{g}^*$; and we shall say that $\varphi$ defines a $\mathrm{G}_2$-structure on $\mathfrak{g}$. We say that a $\mathrm{G}_2$-structure on $\mathfrak{g}$ is *calibrated* if $\varphi$ is closed, i.e.

$$d\varphi = 0,$$

where $d$ denotes the Chevalley-Eilenberg differential on $\mathfrak{g}^*$. If $\Gamma$ is a discrete subgroup of $G$, a $\mathrm{G}_2$-structure on $\mathfrak{g}$ induces a $\mathrm{G}_2$-structure on the quotient $\Gamma\backslash G$. Moreover, in [18] it is proved that if $\mathfrak{g}$ is nilpotent with rational structure constants, then the associated simply connected Lie group $G$ admits a uniform discrete subgroup $\Gamma$. Therefore, a $\mathrm{G}_2$-structure on $\mathfrak{g}$ determines a $\mathrm{G}_2$-structure on the compact manifold $\Gamma\backslash G$, which is called a compact nilmanifold; and if $\mathfrak{g}$ has a calibrated $\mathrm{G}_2$-structure, the $\mathrm{G}_2$-structure on $\Gamma\backslash G$ is also calibrated.

In order to show obstructions to the existence of a calibrated $\mathrm{G}_2$ form on a Lie algebra $\mathfrak{g}$, let us consider first a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. If $\varphi$ is a $\mathrm{G}_2$ form on $\mathfrak{g}$, and the decomposition is orthogonal with respect to the underlying metric, then

$$\varphi = \omega \wedge \eta + \psi^+,$$

where $\omega, \psi^+$ are forms on $\mathfrak{h}$ and $\eta$ generates the dual of the ideal $\mathbb{R}$. The pair $(\omega, \psi^+)$ defines an SU(3)-structure on $\mathfrak{h}$. The condition that $\varphi$ is closed is equivalent to both $\omega$ and $\psi^+$ being closed; this means that the SU(3)-structure is *symplectic half-flat*. There are exactly three nilpotent Lie algebras of dimension six that admit a symplectic half-flat structure, classified in [8]. So, if we focus our attention on decomposable nilpotent Lie algebras, there are at least three 7-dimensional Lie algebras with a calibrated $\mathrm{G}_2$-structure; we will see that these are all.

More generally, every 7-dimensional nilpotent Lie algebra fibres over a nilpotent Lie algebra of dimension six. In fact if $\xi$ is in the center of $\mathfrak{g}$, then the quotient $\mathfrak{g}/\operatorname{Span}\{\xi\}$ has a unique Lie algebra structure that makes the projection map

$$\mathfrak{g} \to \frac{\mathfrak{g}}{\operatorname{Span}\{\xi\}}$$

a Lie algebra morphism. Moreover, due to the nilpotency assumption every epimorphism $\mathfrak{g} \to \mathfrak{h}$, with $\mathfrak{h}$ of dimension six, is of this form. Using the pullback, we can identify forms on the quotient with *basic forms* on $\mathfrak{g}$; in this setting, $\alpha$ is *basic* if $\xi \lrcorner \alpha = 0$.

Given a $\mathrm{G}_2$-structure on $\mathfrak{g}$ with associated 3-form $\varphi$ and a nonzero vector $\xi$ in the center, let $\eta = \xi^\flat$; then we can write

$$\varphi = \omega \wedge \eta + \psi^+, \quad \xi \lrcorner \omega = 0 = \xi \lrcorner \psi^+,$$

and up to a normalization coefficient the forms $(\omega, \psi^+)$ define an SU(3)-structure on the six-dimensional quotient (see also [1]). In analogy with the case of a circle bundle, we shall think of $\eta$ as a connection form, and $d\eta$ as the curvature.

**Proposition 1.** *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with a calibrated $\mathrm{G}_2$-structure and a non-trivial center. If $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra epimorphism with kernel contained in the center, and $\mathfrak{h}$ of dimension six, then $\mathfrak{h}$ admits a symplectic form $\omega$, and the curvature form is in the kernel of*

$$H^2(\mathfrak{h}^*) \xrightarrow{\cdot\wedge\omega} H^4(\mathfrak{h}^*). \tag{2}$$

*If the curvature form is exact on $\mathfrak{h}$, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}$ as Lie algebras.*

*Proof.* Write

$$\varphi = \pi^*\omega \wedge \eta + \pi^*\psi^+$$

where $(\omega, \psi^+)$ are forms on $\mathfrak{h}$. Since $d$ commutes with the pullback,

$$0 = d\varphi = d\pi^*\omega \wedge \eta + \pi^*\omega \wedge d\eta + \pi^* d\psi^+,$$

where $\pi^* d\omega$, $d\eta$ and $\pi^* d\psi^+$ are basic. Thus $\omega$ is a symplectic form and $d\eta$ is in the kernel of (2).

Now suppose that $d\eta$ is exact on $\mathfrak{h}$. Then, the epimorphism $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is trivial, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. More precisely, we can choose a different, closed connection form $\tilde\eta$, and $\mathfrak{g} = \ker\tilde\eta \oplus \ker\pi$ is a direct sum of Lie algebras; by construction, $\ker\tilde\eta$ is isomorphic to $\mathfrak{h}$. ∎

*Remark.* In the previous Proposition, we must notice that when the curvature form is zero, $(\omega, \psi^+)$ is a symplectic half-flat structure on $\mathfrak{h}$. Therefore, if $\mathfrak{h}$ is nilpotent, by [8] $\mathfrak{h}$ is one of

$$(0,0,0,0,0,0), \quad (0,0,0,0,12,13), \quad (0,0,0,12,13,23).$$

With notation from [19],$(0,0,0,0,12,13)$ represents a the Lie algebra with a fixed basis $e^1, \dots e^6$ of $\mathfrak{g}^*$, satisfying

$$de^1 = 0 = de^3 + de^3 = de^4, \quad de^5 = e^{12}, de^6 = e^{13}.$$

*Remark.* Another obstruction to the existence of a calibrated $G_2$-structure on a nilpotent Lie algebra is given by the condition $b_3 > 0$. Indeed, if $\varphi$ is a closed $G_2$ form on a nilpotent Lie algebra $\mathfrak{g}$, and $X$ is a nonzero vector in the center of $\mathfrak{g}$, then $\mathcal{L}_X\varphi = 0$, so $X \lrcorner \varphi$ is closed. If $\varphi$ were exact, say $\varphi = d\beta$, then the 7-form

$$(X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \varphi = d((X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \beta)$$

would also be exact, hence zero, which is absurd. On the other hand, $b_3$ is always positive on a nilpotent Lie algebra of dimension seven.

Proposition 1 motivates the following definition. We say that a 6-dimensional Lie algebra $\mathfrak{h}$ satisfies the *2-Lefschetz property* if, for every symplectic structure on $\mathfrak{h}$, the map (2) is an isomorphism. This condition holds trivially when $\mathfrak{h}$ has no symplectic structure, namely when $\mathfrak{h}$ is one of

$$\begin{array}{ll} (0,0,0,12,23,14+35), & (0,0,0,12,23,14-35), \\ (0,0,0,12,13,14+35), & (0,0,0,0,12,15+34), \\ (0,0,0,0,0,12+34), & (0,0,12,13,14+23,34+52), \\ (0,0,12,13,14,34+52), & (0,0,0,12,14,24). \end{array}$$

It is well known [2] that if $(\mathfrak{h}, \omega)$ is a 6-dimensional, symplectic nilpotent Lie algebra, the map

$$H^1(\mathfrak{h}^*) \xrightarrow{\cdot \wedge \omega^2} H^5(\mathfrak{h}^*).$$

is not surjective. However, in the next proposition, we prove that some of those Lie algebras satisfy the 2-Lefschetz property.

**Proposition 2.** *Among 6-dimensional nilpotent Lie algebras with a symplectic structure, those that satisfy the 2-Lefschetz property are*

$$(0,0,0,0,0,0); \quad (0,0,12,13,23,14); \quad (0,0,12,13,23,14+25).$$

*Proof.* In the abelian case, the bilinear map

$$H^2 \otimes H^2 \to H^4$$

induced by the wedge product is non-degenerate, in the sense that for every nonzero $\beta \in H^2$, the induced linear map $\cdot \wedge \beta \colon H^2 \to H^4$ is an isomorphism.

For the second Lie algebra, the cohomology class of a generic symplectic form is represented by

$$\omega = \lambda_1 e^{16} + \lambda_2(e^{15} + e^{24}) + \lambda_3 e^{25} + \lambda_4(e^{34} - e^{26});$$

non-degeneracy implies $\lambda_4 \neq 0$. The map $H^2 \to H^4$ of (2) is represented by the matrix

$$\begin{pmatrix} \lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\ \lambda_4 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & -2\lambda_4 \\ 0 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$$

which is invertible by the assumption $\lambda_4 \neq 0$.

Similarly, for the last Lie algebra

$$\omega = \lambda_1 e^{14} + \lambda_2(e^{15} + e^{24}) - \lambda_3(e^{26} - e^{34}) + \lambda_4(e^{16} + e^{35}).$$

The map (2) is represented by

$$\begin{pmatrix} \lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\ -\lambda_4 & 2\lambda_3 & 2\lambda_2 & -\lambda_1 \\ 0 & 0 & -2\lambda_3 & 2\lambda_4 \\ 0 & 0 & -\lambda_4 & -\lambda_3 \end{pmatrix},$$

which is invertible unless $\lambda_4^2 + \lambda_3^2 = 0$, which makes $\omega$ degenerate.

For all but three of the remaining Lie algebras, we observe that the bilinear map

$$H^2 \otimes H^2 \to H^4$$

is degenerate in the sense that, for every nonzero $\beta \in H^2$, the map

$$\alpha \to \alpha \wedge \beta, \quad H^2 \to H^4$$

is non-injective. The three exceptions are

$$(0, 0, 12, 13, 23, 14 - 25), \quad (0, 0, 0, 12, 13, 23), \quad (0, 0, 0, 0, 0, 12).$$

However, either Lie algebra has a symplectic form that makes the map (2) non-injective. In fact, on the Lie algebra $\mathfrak{h}$ defined by the equations $(0, 0, 12, 13, 23, 14-25)$, consider the symplectic form

$$\omega = -e^{16} + e^{15} + e^{35} + e^{34} + e^{24} - e^{26}.$$

Then one can check that $e^{14}+e^{25}+e^{15}+e^{24}$ defines a non-trivial class in $H^2(\mathfrak{h}^*)$, but

$$(e^{14} + e^{25} + e^{15} + e^{24}) \wedge \omega = 2e^{1245} = 2d(e^{146}).$$

Now, on the Lie algebra $(0, 0, 0, 12, 13, 23)$ we consider the symplectic form $\omega = e^{14} + e^{26} + e^{35}$. Then,

$$(-e^{15} - e^{24} + e^{36}) \wedge \omega = d(e^{456});$$

finally, on the Lie algebra $(0, 0, 0, 0, 0, 12)$,

$$(e^{16} + e^{25} + e^{34}) \wedge e^{13} = -de^{356}. \qquad \square$$

In principle, one could try to classify all pairs $(\mathfrak{h},\omega)$, with $\mathfrak{h}$ nilpotent of dimension six and $\omega$ a symplectic form on $\mathfrak{h}$, for which (2) is non-injective. This means that $\omega\wedge\gamma = d\psi^+$, for some $\psi^+\in\Lambda^3\mathfrak{h}^*$ and some closed non-exact 2-form $\gamma\in\Lambda^2\mathfrak{h}^*$. If in addition, $(\omega,\psi^+)$ are compatible in the sense that they define an SU(3)-structure, then declaring $de^7=\gamma$ one obtains a 7-dimensional Lie algebra $\mathfrak{g}$ with a calibrated $G_2$-structure. By Proposition 1, all calibrated $G_2$-structures on indecomposable nilpotent Lie algebras are obtained in this way.

However, these calculations turn out to be difficult (although in one dimension less a similar approach was pursued succesfully in [8]), and for this reason we shall use a different method (see Section 4), starting with Gong's classification of 7-dimensional Lie algebras. In fact, given a Lie algebra, it is straightforward to compute the space of its closed 3-forms. In the spirit of [9], the existence of a calibrated $G_2$-structure puts restrictions on this space. Whilst straightforward, the following result turns out to give an effective obstruction.

**Lemma 3.** *Let $\mathfrak{g}$ be a 7-dimensional nilpotent Lie algebra. If there is a nonzero $X$ in $\mathfrak{g}$ such that $(X\lrcorner\phi)^3=0$ for every closed 3-form on $\mathfrak{g}$, then $\mathfrak{g}$ has no calibrated $G_2$-structure.*

*Proof.* Obvious. □

*Remark.* When $\mathfrak{g}$ fibers over a non-symplectic Lie algebra $\mathfrak{h}$, this obstruction is satisfied automatically. Indeed, suppose $\pi\colon\mathfrak{g}\to\mathfrak{h}$ is a Lie algebra epimorphism; then any closed 3-form on $\mathfrak{g}$ can be written as

$$\pi^*\omega\wedge\eta+\pi^*\psi^+,$$

as in the proof of Proposition 1. So $\omega$ is a closed form on $\mathfrak{h}$; if we assume $\mathfrak{h}$ has no symplectic form, then $\omega^3=0$. Then the condition of Lemma 3 is satisfied with $X$ a generator of $\ker\pi$.

# 3 Decomposable case

In this section we classify the decomposable nilpotent Lie algebras with a calibrated $G_2$-structure. Indeed, we prove:

**Theorem 4.** *Among the 35 decomposable nilpotent Lie algebras of dimension 7, those that have a calibrated $G_2$-structure are*

$$(0,0,0,0,0,0,0),\quad (0,0,0,0,12,13,0),\quad (0,0,0,12,13,23,0).$$

*Proof.* By the remark at the beginning of Section 2, we know that these three Lie algebras have a calibrated $G_2$-structure (see [11] where the second of these Lie algebras was considered). In fact, on the non-abelian Lie algebras $(0,0,0,0,12,13)$ and $(0,0,0,12,13,23)$ we can consider the symplectic half flat structure $(\omega_1,\psi_1^+)$ and $(\omega_2,\psi_2^+)$, respectively, defined by

$$\omega_1=e^{14}+e^{26}+e^{35},\quad \psi_1^+=e^{123}+e^{156}+e^{245}-e^{346},$$

and

$$\omega_2 = e^{16} + 2e^{25} + e^{34}, \quad \psi_2^+ = e^{123} + e^{145} + e^{246} - e^{356}.$$

Using Lemma 3, we can see that the decomposable Lie algebra

$$0, 0, 0, 0, 12, 34, 36$$

has no calibrated $G_2$-structure. Indeed a basis of the space $Z^3$ of the closed 3-forms is given by

$$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236},\\ e^{237}, e^{245}, e^{246}, -e^{126} + e^{345}, e^{346}, e^{347}, e^{127} + e^{356}, e^{367}, e^{467}.$$

Thus $e_7 \lrcorner Z^3$ is the span of $e^{13}, e^{23}, e^{34}, e^{12}, e^{36}, e^{46}$, which contains only degenerate forms.

Since this is the only decomposable nilpotent Lie algebra of dimension seven which does not have the form $\mathfrak{h} \oplus \mathbb{R}$, it remains to prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has a calibrated $G_2$ form, then $\mathfrak{g}$ must be as in the statement.

Clearly, if $\mathfrak{h}$ is one of the eight Lie algebras defined by (2), Proposition 1 implies that the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has no calibrated $G_2$ form.

Also, one can check that none of the five Lie algebras defined by

$$(0, 0, 12, 13, 23, 14, 0), \quad (0, 0, 12, 13, 23, 14 + 25, 0), \quad (0, 0, 12, 13, 23, 14 - 25, 0),\\ (0, 0, 0, 0, 13 + 42, 14 + 23, 0), \quad (0, 0, 0, 0, 12, 14 + 23, 0),$$

has a calibrated $G_2$ form because each of these is a bundle over a non-symplectic Lie algebra of dimension six. Explicitly, the base of the bundle and curvature form are given by

$$\pi\colon (0, 0, 12, 13, 23, 14, 0) \to (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14},$$
$$\pi\colon (0, 0, 12, 13, 23, 14 + 25, 0) \to (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14} + e^{25},$$
$$\pi\colon (0, 0, 12, 13, 23, 14 - 25, 0) \to (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14} - e^{25},$$
$$\pi\colon (0, 0, 0, 0, 13 + 42, 14 + 23, 0) \to (0, 0, 0, 0, 13 + 42, 0), \quad d\eta = e^{14} + e^{23},$$
$$\pi\colon (0, 0, 0, 0, 12, 14 + 23, 0) \to (0, 0, 0, 0, 14 + 23, 0), \quad d\eta = e^{12}.$$

For each of the remaining 18 Lie algebras of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$, listed in Table 1 alongside with a basis of the space of closed 3-forms, one can check that the hypothesis of Lemma 3 is satisfied with $X = e_6$. □

# 4 Indecomposable case

In this section we complete the classification of 7-dimensional nilpotent Lie algebras with a calibrated $G_2$-structure. We have seen that there are exactly three decomposable Lie algebras of this type. In order to discuss the indecomposable

Table 1: Closed 3-forms on decomposable Lie algebras

| | |
|---|---|
| $(0,0,12,13,14+23,24+15,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{136}+e^{145}, e^{146}, e^{147}, e^{234}, e^{235}+e^{136}, e^{237}, -e^{236}+e^{245}, e^{157}+e^{247}, e^{167}+e^{257}, -\frac{1}{2}e^{156}+e^{345}-\frac{1}{2}e^{246}, e^{167}+e^{347}$ |
| $(0,0,0,12,14,15+23,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}-e^{146}, -e^{156}+e^{236}, e^{237}, e^{245}, e^{247}, e^{156}+e^{345}, -e^{167}+e^{347}, -e^{267}+e^{457}$ |
| $(0,0,0,12,14-23,15+34,0)$ | $e^{123}, e^{124}, e^{125}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{126}+e^{145}, e^{146}, e^{147}, e^{234}, e^{235}-e^{126}, e^{237}, e^{245}, e^{247}, e^{236}+e^{345}, e^{347}+e^{157}, e^{357}+e^{167}$ |
| $(0,0,0,12,14,15,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{237}, e^{245}, e^{247}, e^{345}+e^{236}, -e^{267}+e^{457}$ |
| $(0,0,0,12,13,14+23,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}-e^{146}, e^{237}, e^{245}+e^{146}, e^{246}, e^{247}, e^{257}+e^{167}, e^{345}+e^{156}, e^{347}-e^{167}, e^{357}$ |
| $(0,0,0,12,13,24,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}-e^{136}, e^{246}, e^{247}, e^{267}, e^{256}+e^{346}, e^{257}+e^{347}, e^{357}$ |
| $(0,0,0,12,14,15+24,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{136}+e^{235}, e^{237}, e^{245}, e^{246}-e^{156}, e^{247}, e^{257}+e^{167}, e^{236}+e^{345}, e^{457}-e^{267}$ |
| $(0,0,0,12,14,15+23+24,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{136}+e^{146}, e^{147}, e^{157}, e^{234}, e^{136}+e^{235}, e^{237}, e^{245}, e^{236}+e^{246}-e^{156}, e^{247}, e^{236}+e^{345}, e^{347}-e^{257}-e^{167}, e^{457}-e^{267}$ |
| $(0,0,12,13,14,15,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{237}, e^{245}-e^{236}, e^{347}-e^{257}$ |
| $(0,0,12,13,14,23+15,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}-e^{146}, e^{236}-e^{156}, e^{237}, e^{245}-e^{156}, e^{247}+e^{167}, e^{347}-e^{257}$ |
| $(0,0,0,12,13+42,14+23,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{136}, e^{137}, e^{145}, -e^{135}+e^{146}, e^{147}, e^{234}, e^{235}, e^{236}-e^{135}, e^{237}, e^{245}+e^{135}, e^{246}, e^{247}, e^{167}+e^{257}, -e^{157}+e^{267}, -e^{167}+e^{347}$ |
| $(0,0,0,12,14,13+42,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}-e^{136}, e^{236}, e^{237}, e^{245}, e^{246}+e^{136}, e^{247}, e^{257}-e^{167}, e^{267}+e^{347}$ |
| $(0,0,0,12,13+14,24,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}+e^{136}, e^{236}, e^{237}, e^{245}-e^{136}, e^{246}, e^{247}, e^{267}, e^{257}+e^{167}+e^{347}$ |
| $(0,0,0,12,13,14,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{237}, e^{245}+e^{236}, e^{246}, e^{247}, e^{347}+e^{257}, e^{357}$ |
| $(0,0,0,0,0,12,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, e^{257}, e^{267}, e^{345}, e^{347}, e^{357}, e^{457}$ |
| $(0,0,0,0,12,14+25,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{237}, e^{245}, e^{246}, e^{247}, -e^{146}+e^{256}, e^{257}, -e^{236}+e^{345}, e^{347}, e^{457}+e^{267}$ |
| $(0,0,0,0,12,34,0)$ | $e^{123}, e^{124}, e^{125}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{257}, e^{345}-e^{126}, e^{346}, e^{347}, e^{367}, e^{467}$ |
| $(0,0,0,0,12,15,0)$ | $e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{237}, e^{245}, e^{247}, e^{256}, e^{257}, e^{347}$ |

Lie algebras, we refer to Gong's classification in [12]. This list consists of 140 Lie algebras and 9 one-parameter families.

The one-parameter families are the following:

$$\begin{aligned}
147E &= \left(0,0,0,e^{12},e^{23},-e^{13},\lambda e^{26}-e^{15}-(-1+\lambda)e^{34}\right), && \lambda\neq 0,1;\\
1357M &= \left(0,0,e^{12},0,e^{24}+e^{13},e^{14},-(-1+\lambda)e^{34}+e^{15}+e^{26}\lambda\right), && \lambda\neq 0;\\
1357N &= \left(0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}\lambda\right);\\
1357S &= \left(0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{16}+e^{15}+\lambda e^{26}\right), && \lambda\neq 1;\\
12457N &= \left(0,0,e^{12},e^{13},e^{23},e^{24}+e^{15},\lambda e^{25}+e^{26}+e^{34}-e^{35}+e^{16}+e^{14}\right);\\
123457I &= \left(0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}+e^{24},\lambda e^{25}-(-1+\lambda)e^{34}+e^{16}\right);\\
147E1 &= \left(0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-e^{16}\lambda+\lambda e^{25}\right), && \lambda>1;\\
1357QRS1 &= \left(0,0,e^{12},0,e^{13}+e^{24},e^{14}-e^{23},e^{26}\lambda+e^{15}-e^{34}(-1+\lambda)\right), && \lambda\neq 0;\\
12457N2 &= \left(0,0,e^{12},e^{13},e^{23},-e^{14}-e^{25},e^{15}-e^{35}+e^{16}+e^{24}+e^{25}\lambda\right), && \lambda\geq 0.
\end{aligned}$$

Recall that a 3-form of type $\mathrm{G}_2$ has the form (1) with respect to some coframe $e^1,\dots,e^7$; such a coframe identifies the $\mathrm{G}_2$-structure.

**Lemma 5.** *Exactly three of the above Lie algebras admit a calibrated* $\mathrm{G}_2$*-structure. Explicit examples are given in terms of a coframe by*

$$\begin{aligned}
1357N(\lambda=1):\quad & \sqrt{3}(2e^1-e^7-e^6-e^5),\sqrt{3}(e^4+e^3-2e^2-e^6+e^5),2e^3-e^6,2e^5,\\
& -e^3+3e^4+e^5-e^6,2e^3-e^5-e^6+3e^7,-\sqrt{3}e^6;\\
1357S(\lambda=-3):\quad & \sqrt{7}(2e^1+e^2-e^5+e^6),7e^2+3e^5+5e^6,\sqrt{7}(e^3+2e^4-\frac{3}{2}e^7),\\
& 3e^3+\frac{7}{2}e^7,-\sqrt{70}e^6,\sqrt{10}(2e^5+e^6),-2\sqrt{10}e^3.\\
147E1(\lambda=2):\quad & \sqrt{3}(2e^1+e^5-e^2+e^6),3e^2-e^5+e^6,e^3+2e^4,\sqrt{3}(e^3+e^7),\\
& \sqrt{2}(e^6-e^5),\sqrt{6}(e^5+e^6),2\sqrt{2}(e^4-e^3).
\end{aligned}$$

*Proof.* It is straightforward to verify that each coframe in the statement determines a calibrated $\mathrm{G}_2$-structure on the corresponding Lie algebra. Conversely, for each Lie algebra $\mathfrak{g}$ the vector $e_7$ is in the center, and determines an epimorphism on a 6-dimensional Lie algebra $\mathfrak{h}$; we view $de^7$ as the curvature form on $\mathfrak{h}$, and apply Proposition 1.

In the case of $1357M$, the generic element of $H^2(\mathfrak{h}^*)$ is represented by

$$\omega=\lambda_6e^{46}+\lambda_3e^{23}+\lambda_1e^{13}+\lambda_5(e^{15}+e^{34})+\lambda_2e^{16}+\lambda_4(e^{15}+e^{26}).$$

Assume $de^7\wedge\omega$ is exact. Then $\lambda_3,\lambda_6$ are zero, $\lambda_4=-\lambda_5\lambda$ and

$$(\lambda-\lambda^2-1)\lambda_5=0.$$

Since $\lambda^2-\lambda+1$ has no real zeroes, $\lambda_4$ and $\lambda_5$ are zero as well, and therefore $\omega^3$ is zero. So there is no symplectic form in the cohomology class of $\omega$. By

Proposition 1, if a calibrated $\mathrm{G}_2$-structure existed, then $\mathfrak{g}$ would have to be decomposable, which is absurd.

The other cases are similar. ◻

We now turn to the rest of the list, where no parameters appear.

**Lemma 6.** *In Gong's list, only six Lie algebras with no parameters in their definition admit a calibrated* $\mathrm{G}_2$*-structure, which can be expressed in terms of a coframe as follows:*

| | |
|---|---|
| $0,0,12,0,0,13+24,15$ | $e^1,e^2,e^5,e^6,e^3,e^7,e^4$ |
| $0,0,12,0,0,13,14+25$ | $e^1,e^3,e^5,e^7,e^2,e^6,e^4$ |
| $0,0,0,12,13,14,15$ | $e^1,e^2,e^4,e^7,e^5,e^6,e^3$ |
| $0,0,0,12,13,14+23,15$ | $e^2+e^7,e^3+e^6,e^7,e^6,e^5,e^4,e^1$ |
| $0,0,12,13,23,15+24,16+34$ | $e^2+e^4,e^7,e^2,e^5,e^3,e^6,e^1$ |
| $0,0,12,13,23,15+24,16+25+34$ | $\sqrt{3}(2e^2+e^5+e^7),2e^4-3e^5-e^7,\sqrt{3}(e^1-e^3+2e^6),$ $e^1+3e^3,\sqrt{6}e^7,\sqrt{2}(2e^4-e^7),2\sqrt{2}e^1$ |

**Theorem 7.** *Up to isomorphism, there are exactly* 12 *nilpotent Lie algebras that admit a calibrated* $\mathrm{G}_2$*-structure, namely those appearing in Theorem 4, Lemma 5 and Lemma 6.*

*Proof.* We must show that the remaining Lie algebras in Gong's list satisfy one of the two obstructions of Section 2; we do so in the Appendix, where we reproduce Gong's list, and note which obstruction applies to each Lie algebra (as a preference, we try to use Proposition 1 rather than Lemma 3 whenever possible, because the former does not require computing the space of closed 3-forms). ◻

# Appendix

This appendix contains a list of all indecomposable nilpotent Lie algebras of dimension 7, taken from [12], except the 9 one-parameter families that we listed at the beginning of Section 4. Alongside each Lie algebra $\mathfrak{g}$, we give a chosen vector $\xi\in\mathfrak{g}$ which satisfies the conditions of Proposition 1 (when marked with a (P)) or Lemma 3, and the structure constants of the quotient $\mathfrak{g}/\operatorname{Span}\{\xi\}$. The word "resists" marks instead the six Lie algebras that resist the obstructions. Below each Lie algebra, we give a basis of its space of closed 3-forms, except when Proposition 1 applies.

Table 2: Step 2 nilpotent Lie algebras of dimension 7

<table>
<tr><td>$0,0,0,0,12,23,24$</td><td>$e_7$</td><td>$[0,0,0,0,e^{12},e^{23}]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{137}+e^{146},e^{147},$<br>$e^{234},e^{235},e^{236},e^{237},e^{245},e^{246},e^{247},e^{256},e^{257},e^{267},e^{345}+e^{137},$<br>$e^{346},e^{347}$</td></tr>
<tr><td>$0,0,0,0,12,23,34$</td><td>$e_7$</td><td>$[0,0,0,0,e^{12},e^{23}]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146}-e^{127},e^{147},$<br>$e^{234},e^{235},e^{236},e^{237},e^{245},e^{246},e^{247},e^{256},-e^{127}+e^{345},e^{346},e^{347},e^{367}$</td></tr>
<tr><td>$0,0,0,0,12+34,23,24$</td><td>$e_5$</td><td>$[0,0,0,0,e^{23},e^{24}]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137}+e^{125},e^{145},-e^{125}+e^{146},$<br>$e^{147},e^{234},e^{235},e^{236},e^{237},e^{245},e^{246},e^{247},e^{267},-e^{125}+e^{345},e^{346},$<br>$e^{347},-e^{256}+e^{367},-e^{257}+e^{467}$</td></tr>
<tr><td>$0,0,0,0,12+34,13,24$</td><td>$e_7$</td><td>$[0,0,0,0,e^{12}+e^{34},e^{13}]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137}+e^{125},e^{145},e^{146},e^{147},$<br>$e^{234},e^{235},e^{236},e^{237},e^{245},e^{246}+e^{125},e^{247},-e^{125}+e^{345},e^{346},e^{347}$</td></tr>
<tr><td>$0,0,0,0,0,12,14+35$</td><td>$e_6(P)$</td><td>$[0,0,0,0,0,e^{14}+e^{35}]$</td></tr>
<tr><td>$0,0,0,0,0,12+34,15+23$</td><td>$e_7(P)$</td><td>$[0,0,0,0,0,e^{34}+e^{12}]$</td></tr>
<tr><td>$0,0,0,0,0,0,12+34+56$</td><td>$e_7$</td><td>$[0,0,0,0,0,0]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},$<br>$e^{236},e^{245},e^{246},e^{256},e^{345},e^{346},e^{356},e^{456}$</td></tr>
<tr><td>$0,0,0,0,12-34,13+24,14$</td><td>$e_6$</td><td>$[0,0,0,0,e^{12}-e^{34},e^{14}]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{126},e^{127},e^{134},e^{135},e^{136}-e^{125},e^{137},e^{145},e^{146},e^{147},$<br>$e^{234},e^{235},e^{236},e^{125}+e^{237},e^{245},-e^{125}+e^{246},e^{247},e^{125}+e^{345},e^{346},$<br>$e^{347},e^{457}-e^{167},e^{467}+e^{157}$</td></tr>
<tr><td>$0,0,0,0,12-34,13+24,14-23$</td><td>$e_7$</td><td>$[0,0,0,0,e^{12}-e^{34},e^{24}+e^{13}]$</td></tr>
<tr><td colspan="3">$e^{123},e^{124},e^{126},e^{127},e^{134},e^{135},e^{136}-e^{125},e^{137},e^{145},e^{146},-e^{125}+e^{147},$<br>$e^{234},e^{235},e^{236},e^{125}+e^{237},e^{245},-e^{125}+e^{246},e^{247},e^{125}+e^{345},e^{346},e^{347}$</td></tr>
</table>

Table 3: Step 3 nilpotent Lie algebras of dimension 7

| | | |
|---|---|---|
| $0,0,12,0,13,24,14$ | $e_5$ | $[0,0,e^{12},0,e^{24},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{146},e^{147},e^{157},$ $e^{234},e^{235},e^{236},e^{237}+e^{136},e^{245}-e^{136},e^{246},e^{247},e^{345}-e^{156},e^{346}+e^{267},$ $e^{347}+e^{167},e^{467}$ | | |
| $0,0,12,0,13,23,14$ | $e_7(P)$ | $[0,0,e^{12},0,e^{13},e^{23}]$ |
| $0,0,12,0,13+24,23,14$ | $e_6$ | $[0,0,e^{12},0,e^{24}+e^{13},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{137},e^{145},e^{146}+e^{135},e^{147},$ $e^{234},e^{235},e^{236},e^{237}+e^{135},-e^{135}+e^{245},e^{246},e^{247},e^{257}+e^{345}+e^{167},$ $e^{346}+e^{267},e^{347}+e^{157}$ | | |
| $0,0,12,0,0,13+24,15$ | resists | |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{146},e^{147},e^{157},$ $e^{234},e^{235},e^{236},e^{245},-e^{136}+e^{246},-e^{156}+e^{247},e^{256}+e^{237},e^{257},e^{345}-e^{156},$ $-e^{347}-e^{167}+e^{456},e^{457}$ | | |
| $0,0,12,0,0,13,14+25$ | resists | |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{157},$ $e^{234},e^{235},e^{236},e^{245},e^{237}+e^{246},e^{247},e^{256}-e^{137},e^{257}-e^{147},e^{345}+e^{147},$ $-e^{167}+e^{356},e^{457}$ | | |
| $0,0,12,0,0,13+24,25$ | $e_7$ | $[0,0,e^{12},0,0,e^{13}+e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{146},e^{147}+e^{156},e^{157},$ $e^{234},e^{235},e^{236},e^{237},e^{245},-e^{136}+e^{246},e^{247},e^{256}-e^{137},e^{257},e^{345}+e^{147},e^{457}$ | | |
| $0,0,12,0,0,13+24,14+25$ | $e_7$ | $[0,0,e^{12},0,0,e^{13}+e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{146},e^{156}+e^{147},e^{157},$ $e^{234},e^{235},e^{236},e^{237}+e^{136},e^{245},e^{246}-e^{136},e^{247},e^{256}-e^{137},e^{257}-e^{147},$ $e^{345}+e^{147},e^{457}$ | | |
| $0,0,12,0,0,13+45,24$ | $e_7(P)$ | $[0,0,e^{12},0,0,e^{13}+e^{45}]$ |
| $0,0,12,0,0,13+45,15+24$ | $e_7(P)$ | $[0,0,e^{12},0,0,e^{13}+e^{45}]$ |
| $0,0,12,0,0,13+24,45$ | $e_7$ | $[0,0,e^{12},0,0,e^{13}+e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{146},e^{147},-e^{127}+e^{156},e^{157},$ $e^{234},e^{235},e^{236},e^{245},e^{246}-e^{136},e^{247},e^{257},e^{345}-e^{127},e^{456}-e^{137},e^{457}$ | | |
| $0,0,12,0,0,13+14,15+23$ | $e_7$ | $[0,0,e^{12},0,0,e^{14}+e^{13}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{156},$ $e^{234},e^{235},-e^{147}+e^{236},e^{237}-e^{157},e^{245},e^{147}+e^{246},e^{247}+e^{256}+e^{157},$ $e^{257},-e^{247}+e^{345}$ | | |
| $0,0,12,0,0,13+24,15+23$ | $e_7$ | $[0,0,e^{12},0,0,e^{24}+e^{13}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{146},e^{147}+e^{136},$ $e^{234},e^{235},e^{236},-e^{157}+e^{237},e^{245},e^{246}-e^{136},-e^{156}+e^{247},e^{157}+e^{256},$ $e^{257},-e^{156}+e^{345}$ | | |
| $0,0,12,0,0,13,23+45$ | $e_6(P)$ | $[0,0,e^{12},0,0,e^{23}+e^{45}]$ |
| $0,0,12,0,0,13+24,23+45$ | $e_6(P)$ | $[0,0,e^{12},0,0,e^{23}+e^{45}]$ |

| | | |
|---|---|---|
| $0,0,0,12,13,14,15$ | resists | |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{147},$<br>$e^{156},e^{157},e^{167},e^{234},e^{235},e^{245}+e^{236},e^{246},e^{345}+e^{237},e^{346}+e^{256}+e^{247},$<br>$e^{257}+e^{347}+e^{356},e^{357}$ | | |
| $0,0,0,12,13,14,35$ | $e_7$ | $[0,0,0,e^{12},e^{13},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{156},e^{157},$<br>$e^{234},e^{235},e^{237},e^{245}+e^{236},e^{246},e^{345}+e^{127},e^{347}+e^{257},-e^{147}+e^{356},e^{357}$ | | |
| $0,0,0,12,13,14+35,15$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{35}+e^{14}]$ |
| $0,0,0,12,13,14,25+34$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{34}+e^{25}]$ |
| $0,0,0,12,13,14+15,25+34$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{25}+e^{34}]$ |
| $0,0,0,12,13,24+35,25+34$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{24}+e^{35}]$ |
| $0,0,0,12,13,14+15+24+35,25+34$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{35}+e^{24}+e^{15}+e^{14}]$ |
| $0,0,0,12,13,14+24+35,25+34$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{24}+e^{35}+e^{14}]$ |
| $0,0,0,12,13,25+34,35$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{25}+e^{34}]$ |
| $0,0,0,12,13,15+35,25+34$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{34}+e^{25}]$ |
| $0,0,0,12,13,14+35,25+34$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{14}+e^{35}]$ |
| $0,0,0,12,13,14+23,15$ | resists | |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{147},e^{157},$<br>$e^{234},e^{235},-e^{146}+e^{236},-e^{156}+e^{237},e^{245}+e^{146},e^{246},e^{167}+e^{257},e^{156}+e^{345},$<br>$e^{346}+e^{256}+e^{247},-e^{167}+e^{356}+e^{347},e^{357}$ | | |
| $0,0,0,12,13,14+23,35$ | $e_7$ | $[0,0,0,e^{12},e^{13},e^{23}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145},-e^{127}+e^{156},e^{157},$<br>$e^{234},e^{235},-e^{146}+e^{236},e^{237},e^{146}+e^{245},e^{246},e^{127}+e^{345},e^{257}+e^{347},$<br>$-e^{147}+e^{356},e^{357}$ | | |
| $0,0,0,12,13,15+24,23$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{24}+e^{15}]$ |
| $0,0,0,12,13,14+35,15+23$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{35}+e^{14}]$ |
| $0,0,0,12,13,23,25+34$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{25}+e^{34}]$ |
| $0,0,0,12,13,14+23,25+34$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{25}+e^{34}]$ |
| $0,0,0,12,13,14+15+23,25+34$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{25}+e^{34}]$ |
| $0,0,12,0,0,0,13+24+56$ | $e_7$ | $[0,0,e^{12},0,0,0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},$<br>$e^{236},e^{245},e^{246},e^{256},-e^{157}+e^{345},-e^{167}+e^{346},e^{356}-e^{127},e^{456}$ | | |
| $0,0,0,12,13,0,16+25+34$ | $e_7$ | $[0,0,0,e^{12},e^{13},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},$<br>$e^{236},e^{245}+e^{127},e^{246},e^{256}-e^{237},e^{345}-e^{137},e^{346}+e^{237},e^{356}$ | | |
| $0,0,0,12,13,0,14+26+35$ | $e_7$ | $[0,0,0,e^{12},e^{13},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},$<br>$e^{236},e^{237}+e^{245},e^{246},e^{137}+e^{256},e^{127}+e^{345},-e^{137}+e^{346},e^{356},e^{157}+e^{456}-e^{367}$ | | |
| $0,0,0,12,23,-13,15+26+16-2*34$ | $e_7$ | $[0,0,0,e^{12},e^{23},-e^{13}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},\frac{1}{2}e^{127}+e^{145},e^{146},$<br>$e^{156}-e^{137},e^{234},e^{235},e^{236},e^{245},e^{246}+\frac{1}{2}e^{127},e^{256}-e^{137}-e^{237},$<br>$e^{345}+e^{137}+e^{237},e^{346}-e^{137},e^{356}$ | | |
| $0,0,0,0,12,34,15+36$ | $e_7$ | $[0,0,0,0,e^{12},e^{34}]$ |
| $e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{234},e^{235},e^{236},$<br>$e^{245},e^{246},e^{345}-e^{126},e^{346},e^{347}-e^{156},e^{356}+e^{127},-e^{157}+e^{367}$ | | |

| | | |
|---|---|---|
| $0,0,0,0,12,34,15+24+36$ | $e_7$ | $[0,0,0,0,e^{12},e^{34}]$ |
| $e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137}+e^{126},e^{145},e^{146},e^{234},e^{235},$ $e^{236},e^{245},e^{246},e^{345}-e^{126},e^{346},e^{347}-e^{156},e^{356}+e^{127}$ | | |
| $0,0,0,0,12,14+23,16-35$ | $e_7$ | $[0,0,0,0,e^{12},e^{14}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145},e^{156}+e^{127},e^{157},$ $e^{234},e^{235},-e^{146}+e^{236},e^{245},e^{246},-e^{146}+e^{345},e^{346},e^{147}+e^{237}+e^{356}$ | | |
| $0,0,0,0,12,14+23,16+24-35$ | $e_7$ | $[0,0,0,0,e^{12},e^{23}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{137}+e^{146},e^{156}+e^{127},$ $e^{234},e^{235},e^{236}+e^{137},e^{245},e^{246},e^{157}+e^{256},e^{345}+e^{137},e^{346},e^{356}+e^{237}+e^{147}$ | | |
| $0,0,12,0,0,13+14+25,15+23$ | $e_7$ | $[0,0,e^{12},0,0,e^{25}+e^{14}+e^{13}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{156},e^{157}+e^{136},$ $e^{234},e^{235},-e^{147}+e^{236},e^{136}+e^{237},e^{245},e^{147}+e^{246},e^{146}+e^{247},e^{256}-e^{146}-e^{136},$ $e^{257},e^{345}+e^{146}$ | | |
| $0,0,0,12,13,14,24+35$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{24}+e^{35}]$ |
| $0,0,0,12,13,24-35,25+34$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},-e^{35}+e^{24}]$ |
| $0,0,0,12,13,14+24-35,25+34$ | $e_7(P)$ | $[0,0,0,e^{12},e^{13},e^{24}-e^{35}+e^{14}]$ |
| $0,0,0,12,13,23,24+35$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{35}+e^{24}]$ |
| $0,0,0,12,13,14+23,24+35$ | $e_6(P)$ | $[0,0,0,e^{12},e^{13},e^{24}+e^{35}]$ |
| $0,0,0,12,13,0,16+24+35$ | $e_7$ | $[0,0,0,e^{12},e^{13},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},$ $e^{236},e^{245}-e^{137},e^{246},e^{256}-e^{237},e^{127}+e^{345},e^{237}+e^{346},e^{356}$ | | |
| $0,0,0,0,13+24,14-23,15+26$ | $e_7$ | $[0,0,0,0,e^{13}+e^{24},e^{14}-e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},-e^{135}+e^{146},e^{234},e^{235},$ $e^{135}+e^{236},e^{237}-e^{147}+e^{156},-e^{135}+e^{245},e^{246},e^{137}+e^{247}+e^{256},e^{257}-e^{167},$ $e^{345},e^{346}$ | | |
| $0,0,0,0,13+24,14-23,15+26+24$ | $e_7$ | $[0,0,0,0,e^{24}+e^{13},e^{14}-e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},e^{146}-e^{135},e^{234},e^{235},$ $e^{135}+e^{236},e^{156}-e^{147}+e^{237},-e^{135}+e^{245},e^{246},e^{256}+e^{137}-e^{135}+e^{247},$ $e^{257}-e^{167}-e^{147},e^{345},e^{346}$ | | |

Table 4: Step 4 nilpotent Lie algebras of dimension 7

| | | |
|---|---|---|
| $0,0,12,13,0,14,15$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{147},$ $e^{156},e^{157},e^{167},e^{234},e^{235},-e^{237}+e^{245},e^{257},e^{256}+e^{345}$ | | |
| $0,0,12,13,0,25,14$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{25}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{147},e^{156},e^{157},$ $e^{234},e^{235},e^{236},e^{136}+e^{245},e^{256},-e^{146}+e^{257},e^{146}+e^{345}$ | | |
| $0,0,12,13,0,14+25,15$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{147},e^{156},e^{157},$ $e^{234},e^{235},e^{136}+e^{237},e^{136}+e^{245},e^{256}-e^{146},e^{257},e^{345}+e^{146},e^{357}+e^{167}$ | | |

| | | |
|---|---|---|
| $0,0,12,13,0,14+23+25,15$ | $e_6$ | $[0,0,e^{12},e^{13},0,e^{15}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},$ $e^{147},e^{136}+e^{156},e^{157},e^{234},e^{235},e^{237}+e^{136},e^{245}+e^{136},$ $e^{256}+e^{236}-e^{146},e^{257},e^{345}-e^{236}+e^{146},e^{247}-e^{236}+e^{357}+e^{167}+e^{146}$ | | |
| $0,0,12,13,0,23+25,14$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{23}+e^{25}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{147},e^{156}+e^{136},$ $e^{157},e^{234},e^{235},e^{236},e^{136}+e^{245},e^{256},e^{257}-e^{146}+e^{237},e^{345}+e^{146}-e^{237}$ | | |
| $0,0,12,13,0,14+23,15$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{23}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{147},e^{157},e^{234},e^{235},$ $-e^{146}+e^{236},-e^{156}+e^{237},e^{245}-e^{156},e^{247}+e^{256}+e^{167},e^{257},e^{345}-e^{247}-e^{167}$ | | |
| $0,0,12,13,0,15+23,14$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{15}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{147},e^{157},$ $e^{234},e^{235},-e^{156}+e^{236},-e^{146}+e^{237},-e^{156}+e^{245},e^{167}+e^{247},e^{256},e^{345}+e^{257}$ | | |
| $0,0,12,13,0,23,14+25$ | $e_7(P)$ | $[0,0,e^{12},e^{13},0,e^{23}]$ |
| $0,0,12,13,0,14+23,25$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{14}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{137}+e^{156},e^{157},$ $e^{234},e^{235},-e^{146}+e^{236},e^{237},e^{137}+e^{245},-e^{147}+e^{256},e^{257},e^{147}+e^{345}$ | | |
| $0,0,12,13,0,14+23,23+25$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{23}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{156}+e^{137},e^{157}+e^{137},$ $e^{234},e^{235},-e^{146}+e^{236},e^{237},e^{137}+e^{245},e^{146}-e^{147}+e^{256},e^{257},-e^{146}+e^{147}+e^{345}$ | | |
| $0,0,12,13,0,15+23,14+25$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{15}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{156}+e^{137},e^{157},e^{234},e^{235},e^{236}+e^{137},$ $-e^{146}+e^{237},e^{245}+e^{137},e^{256},-e^{147}+e^{257},e^{345}+e^{147},e^{356}-e^{167}-e^{247}$ | | |
| $0,0,12,13,23,14+25,15+24$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{23},e^{14}+e^{25}]$ |
| $0,0,12,13,23,24+15,14$ | $e_6(P)$ | $[0,0,e^{12},e^{13},e^{23},e^{14}]$ |
| $0,0,0,12,14+23,23,15-34$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{127},e^{146}+e^{127},e^{147},e^{234},$ $e^{235}+e^{127},e^{236},e^{245},e^{246},e^{345}+e^{237},-e^{156}+e^{346}+e^{237},e^{167}-\frac{1}{2}e^{157}-\frac{1}{2}e^{347}+e^{356}$ | | |
| $0,0,0,12,14+23,13,15-34$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},e^{13}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{127}+e^{145},e^{146},e^{147},$ $e^{234},e^{127}+e^{235},e^{236},e^{245},-e^{127}+e^{246},e^{237}+e^{345},-e^{156}+e^{346},e^{167}+e^{356}$ | | |
| $0,0,0,12,14+23,24,15-34$ | $e_7$ | $[0,0,0,e^{12},e^{14}+e^{23},e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}-e^{127},e^{137},e^{145}+e^{127},e^{146},$ $e^{147},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{345}+e^{237},e^{247}+e^{346}-e^{156}$ | | |
| $0,0,0,12,14+23,13+24,15-34$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},e^{24}+e^{13}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}-e^{127},e^{137},e^{145}+e^{127},e^{146},$ $e^{147},e^{234},e^{127}+e^{235},e^{236},e^{245},e^{246}-e^{127},e^{345}+e^{237},e^{346}-e^{156}+e^{247}$ | | |
| $0,0,12,13,0,0,14+56$ | $e_7$ | $[0,0,e^{12},e^{13},0,0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{157},e^{167},$ $e^{234},e^{235},e^{236},e^{256},e^{257}+e^{345},e^{346}+e^{267},-e^{127}+e^{356},e^{456}-e^{137},e^{567}-e^{147}$ | | |
| $0,0,12,13,0,0,23+14+56$ | $e_7$ | $[0,0,e^{12},e^{13},0,0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},e^{236},-e^{157}+e^{245},$ $-e^{167}+e^{246},e^{256},e^{257}+e^{345},e^{346}+e^{267},-e^{127}+e^{356},-e^{137}+e^{456}$ | | |
| $0,0,0,12,14+23,0,15+26-34$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{127},e^{146},-e^{137}+e^{156},$ $e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{256}+e^{147},e^{345}+e^{237},e^{346}-e^{137},e^{356}+e^{167}$ | | |

| | | |
|---|---|---|
| $0,0,0,12,14+23,0,15+36-34$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{146},e^{127}+e^{156}+e^{145},-e^{147}+e^{167},$ $e^{234},e^{235}-e^{145},e^{236},e^{245},e^{246},e^{237}+e^{345},e^{346}+e^{127}+e^{145},e^{147}+e^{356}$ | | |
| $0,0,0,12,14+23,0,15+24+36-34$ | $e_7$ | $[0,0,0,e^{12},e^{14}+e^{23},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{137},e^{146},e^{156}+e^{127}-e^{137},$ $e^{234},e^{235}+e^{137},e^{236},e^{245},e^{246},-e^{167}+e^{147}+e^{256},$ $e^{345}+e^{237},e^{346}+e^{127}-e^{137},e^{147}+e^{356}$ | | |
| $0,0,12,0,23,24,16+25+34$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},e^{235},e^{236},$ $e^{147}+e^{237}+e^{156},e^{245},e^{246},e^{256},e^{147}+e^{345},-e^{247}+e^{346},e^{456}+e^{267}$ | | |
| $0,0,12,0,23,24,25+46$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145}+e^{136},e^{146},e^{234},e^{235},e^{236},$ $e^{245},e^{246},e^{247},e^{256},e^{267},e^{147}+e^{345},-e^{127}+e^{346},e^{237}+e^{456},e^{467}-e^{257}$ | | |
| $0,0,12,0,23,24,13+25-46$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{145},e^{146},e^{234},e^{235},e^{236},e^{245},$ $e^{246},e^{247}-e^{136},e^{256},-e^{147}+e^{267}-e^{156},e^{147}+e^{345},e^{127}+e^{346},-e^{237}+e^{456}$ | | |
| $0,0,12,0,23,14,16+25$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},$ $e^{156}+e^{237},e^{245},e^{246},-e^{147}+e^{256},-e^{167}+e^{257},e^{147}+e^{345},e^{346}-e^{247}$ | | |
| $0,0,12,0,23,14,16+25+26-34$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{127}+e^{145},e^{146},e^{234},e^{235},e^{127}+e^{236},$ $e^{156}+e^{237},e^{245},e^{246},e^{256}-e^{247}-e^{147},e^{345}+e^{247}+e^{147},e^{346}-e^{247}$ | | |
| $0,0,12,0,23,14,25+46$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},$ $e^{245},e^{246},e^{247},e^{256}-e^{147},e^{345}+e^{147},e^{346}-e^{127},e^{237}+e^{456},-e^{257}+e^{467}$ | | |
| $0,0,12,0,23,14,13+25+46$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},$ $e^{245},e^{246},e^{145}+e^{247},e^{256}-e^{147},e^{147}+e^{345},-e^{127}+e^{346},e^{237}+e^{456}$ | | |
| $0,0,12,0,13+24,14,15+23+1/2*(26+34)$ | $e_7$ | $[0,0,e^{12},0,e^{13}+e^{24},e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},2e^{127}+e^{135},e^{136},e^{145},e^{146},4e^{127}-2e^{147}+e^{156},$ $e^{234},e^{235},-2e^{127}+e^{236},2e^{127}+e^{245},e^{246},2e^{137}+e^{247}+e^{256},-e^{247}+e^{345},$ $-4e^{127}+e^{346}+2e^{147},e^{456}+4e^{137}+2e^{167}$ | | |
| $0,0,12,0,13+24,23,16+25$ | $e_7$ | $[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},e^{146}+e^{135},e^{234},e^{235},e^{236},-e^{135}+e^{245},$ $e^{246},e^{156}+e^{137}+e^{247},e^{256}-e^{237},e^{267},e^{345}+e^{147},e^{346}+e^{156}+e^{137}$ | | |
| $0,0,12,0,13+24,23,15+26+34$ | $e_7$ | $[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{127}+e^{135},e^{136},e^{145},e^{146}-e^{127},e^{234},$ $e^{235},e^{236},e^{237}-e^{156}+e^{147},e^{127}+e^{245},e^{246},e^{256}+e^{137},-e^{247}+e^{345},e^{346}+e^{147}$ | | |
| $0,0,12,0,13,23+24,15+26$ | $e_7$ | $[0,0,e^{12},0,e^{13},e^{23}+e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{136}+e^{146},e^{234},e^{235},e^{236},-e^{136}+e^{245},$ $e^{246},e^{247}+e^{237}-e^{156},e^{256}+e^{137},-e^{157}+e^{267},e^{237}-e^{156}+e^{345},e^{147}+e^{346}$ | | |
| $0,0,12,0,13,23+24,16+25+34$ | $e_7$ | $[0,0,e^{12},0,e^{13},e^{24}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{145},-e^{127}+e^{146},e^{137}+e^{147}+e^{156},$ $e^{234},e^{235},e^{236},e^{127}+e^{245},e^{246},e^{256}-e^{237},e^{345}+e^{147},e^{346}-e^{247}$ | | |

| | | |
|---|---|---|
| $0,0,12,13,23,14-25,15+24$ | $e_7$ | $[0,0,e^{12},e^{13},e^{23},-e^{25}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145}+e^{137},e^{147},e^{234},e^{235},$ $e^{236}+e^{137},-e^{136}+e^{237},-e^{136}+e^{245},e^{156}+e^{246},e^{247}+e^{157}-2e^{146},$ $e^{256}+e^{146},e^{257},-e^{146}+e^{345}$ | | |
| $0,0,0,12,14+23,13-24,15-34$ | $e_7$ | $[0,0,0,e^{12},e^{14}+e^{23},e^{13}-e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{137},e^{145}+e^{127},e^{146},$ $e^{147},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246}-e^{127},e^{345}+e^{237},-e^{247}+e^{346}-e^{156}$ | | |
| $0,0,12,0,23,24,13+25+46$ | $e_7$ | $[0,0,e^{12},0,e^{23},e^{24}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145}+e^{136},e^{146},e^{234},e^{235},e^{236},$ $e^{245},e^{246},e^{247}-e^{136},e^{256},e^{267}-e^{147}-e^{156},e^{345}+e^{147},e^{346}-e^{127},e^{237}+e^{456}$ | | |
| $0,0,12,0,13+24,23,15+34-26$ | $e_7$ | $[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135}+e^{127},e^{136},e^{145},e^{146}-e^{127},e^{234},$ $e^{235},e^{236},-e^{156}-e^{147}+e^{237},e^{245}+e^{127},e^{246},e^{256}-e^{137},e^{345}-e^{247},e^{346}-e^{147}$ | | |
| $0,0,12,0,13,23+24,15-26$ | $e_7$ | $[0,0,e^{12},0,e^{13},e^{24}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{146}+e^{136},e^{234},e^{235},e^{236},e^{245}-e^{136},$ $e^{246},-e^{156}+e^{237}+e^{247},-e^{137}+e^{256},e^{157}+e^{267},-e^{156}+e^{345}+e^{237},-e^{147}+e^{346}$ | | |

Table 5: Step 5 nilpotent Lie algebras of dimension 7

| | | |
|---|---|---|
| $0,0,12,13,14,15,23$ | $e_6(P)$ | $[0,0,e^{12},e^{13},e^{14},e^{23}]$ |
| $0,0,12,13,14,25-34,23$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{14},e^{25}-e^{34}]$ |
| $0,0,12,13,14,15,25-34$ | $e_6(P)$ | $[0,0,e^{12},e^{13},e^{14},e^{25}-e^{34}]$ |
| $0,0,12,13,14,15+23,25-34$ | $e_6(P)$ | $[0,0,e^{12},e^{13},e^{14},-e^{34}+e^{25}]$ |
| $0,0,12,13,14+23,15+24,23$ | $e_6(P)$ | $[0,0,e^{12},e^{13},e^{23}+e^{14},e^{23}]$ |
| $0,0,12,13,14+23,25-34,23$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{14}+e^{23},e^{25}-e^{34}]$ |
| $0,0,12,13,14+23,15+24,25-34$ | $e_6(P)$ | $[0,0,e^{12},e^{13},e^{23}+e^{14},-e^{34}+e^{25}]$ |
| $0,0,12,13,14,0,15+26$ | $e_7$ | $[0,0,e^{12},e^{13},e^{14},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{167},$ $e^{234},e^{236},-e^{237}+e^{245},e^{246}+e^{137},e^{256}+e^{147},-e^{157}+e^{267},e^{147}+e^{346}$ | | |
| $0,0,12,13,14,0,15+23+26$ | $e_7$ | $[0,0,e^{12},e^{13},e^{14},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{137}+e^{167},e^{234},e^{236},$ $-e^{237}+e^{245},e^{137}+e^{246},e^{147}+e^{256}-e^{235},-e^{157}+e^{267}+e^{237},e^{147}-e^{235}+e^{346}$ | | |
| $0,0,12,13,14,0,16+25-34$ | $e_7$ | $[0,0,e^{12},e^{13},e^{14},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{127}+e^{235},$ $e^{236},e^{137}+e^{245},-e^{237}+e^{246},e^{147}+e^{345},e^{346}-e^{256}$ | | |
| $0,0,12,13,14+23,0,15+24+26$ | $e_7$ | $[0,0,e^{12},e^{13},e^{14}+e^{23},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{146},-e^{137}+e^{156}-e^{145},$ $e^{167}+e^{147},e^{234},e^{235}-e^{145},e^{236},-e^{237}+e^{245},e^{137}+e^{145}+e^{246},e^{147}+e^{256},$ $-\frac{1}{2}e^{157}-\frac{1}{2}e^{247}+\frac{1}{2}e^{267}+e^{345},e^{147}+e^{346}$ | | |

| | | |
|---|---|---|
| $0,0,12,13,14+23,0,16+25-34$ | $e_7$ | $[0,0,e^{12},e^{13},e^{23}+e^{14},0]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{127},e^{146},e^{234},e^{127}+e^{235},$<br>$e^{236},e^{237}+e^{156},e^{137}+e^{245},e^{246}+e^{156},e^{147}+e^{345},e^{346}-e^{256}$ | | |
| $0,0,12,13,14,23,15+26$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{14},e^{23}]$ |
| $0,0,12,13,14,23,16+24+25-34$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{14},e^{23}]$ |
| $0,0,12,13,14,23,15+25+26-34$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{14},e^{23}]$ |
| $0,0,12,13,0,14+25,16+35$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{137},e^{145},e^{156},e^{157},$<br>$e^{234},e^{235},e^{245}-e^{127},-e^{237}+e^{246},e^{256}-e^{146},$<br>$e^{345}+e^{146},-e^{257}+e^{356}-e^{147},-\frac{1}{2}e^{167}-\frac{1}{2}e^{357}+e^{456}$ | | |
| $0,0,12,13,0,14+25,16+25+35$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{137}+e^{127},e^{145},e^{156},$<br>$e^{157},e^{234},e^{235},e^{245}-e^{127},e^{246}-e^{237},e^{256}-e^{146},e^{345}+e^{146},$<br>$-e^{257}+e^{356}+e^{146}-e^{147},\frac{1}{2}e^{257}-\frac{1}{2}e^{357}+e^{456}-\frac{1}{2}e^{167}$ | | |
| $0,0,12,13,0,14+25,26-34$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{156},e^{234},e^{235},e^{236}+e^{127},$<br>$e^{237},e^{245}+e^{136},e^{246}+e^{137},e^{256}-e^{146},e^{146}+e^{345},e^{147}+e^{257}+e^{346}$ | | |
| $0,0,12,13,0,14+25,15+26-34$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{14}+e^{25}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{156},e^{234},e^{235},e^{127}+e^{236},$<br>$e^{136}+e^{237},e^{136}+e^{245},e^{137}+e^{246},e^{256}-e^{146},e^{345}+e^{146},e^{257}+e^{346}+e^{147}$ | | |
| $0,0,12,13,0,14+23+25,16+24+35$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{14}+e^{25}+e^{23}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145},e^{156}-e^{127},e^{157}+e^{137}+e^{146},$<br>$e^{234},e^{235},e^{137}+e^{236},e^{245}-e^{127},-e^{237}+e^{246},e^{256}-e^{137}-e^{146},$<br>$e^{345}+e^{137}+e^{146},-e^{257}-e^{147}+e^{356}$ | | |
| $0,0,12,13,0,14+23+25,26-34$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{23}+e^{14}+e^{25}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{156}+e^{136},e^{234},e^{235},e^{236}+e^{127},$<br>$e^{237},e^{245}+e^{136},e^{246}+e^{137},e^{256}-e^{146}-e^{127},e^{345}+e^{146}+e^{127},e^{346}+e^{257}+e^{147}$ | | |
| $0,0,12,13,0,14+23+25,15+26-34$ | $e_7$ | $[0,0,e^{12},e^{13},0,e^{23}+e^{25}+e^{14}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{136}+e^{156},e^{234},e^{235},e^{127}+e^{236},e^{136}+e^{237},$<br>$e^{136}+e^{245},e^{137}+e^{246},-e^{127}+e^{256}-e^{146},e^{345}+e^{127}+e^{146},e^{346}+e^{147}+e^{257}$ | | |
| $0,0,12,13,23,15+24,16+34$ | resists | |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{137},-e^{127}+e^{145},e^{146},$<br>$e^{147},e^{234},e^{235},e^{245}-e^{236},e^{246}-e^{156}-2e^{237},e^{256},e^{345}-e^{156}-e^{237},$<br>$e^{346}-e^{247}-e^{157},e^{267}-e^{357}+e^{456}$ | | |
| $0,0,12,13,23,15+24,16+25+34$ | resists | |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},$<br>$e^{235},e^{236}+e^{137},e^{237}+e^{147}+e^{156},e^{137}+e^{245},2e^{147}+e^{156}+e^{246},e^{256},$<br>$e^{345}+e^{147},e^{346}-e^{247}-e^{157},e^{456}-e^{357}+e^{267}$ | | |
| $0,0,12,13,23,15+24,16+14+25+34$ | $e_7$ | $[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},$<br>$e^{235},e^{236}+e^{137},e^{237}+e^{156}+e^{147}-e^{127},e^{245}+e^{137},e^{246}+e^{156}+2e^{147},$<br>$e^{256},e^{345}+e^{147},e^{346}-e^{247}-e^{157}$ | | |

| | | |
|---|---|---|
| $0,0,12,13,23,15+24,16+14+34$ | $e_7$ | $[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{137},-e^{127}+e^{145},e^{146},$<br>$e^{147},e^{234},e^{235},-e^{236}+e^{245},2e^{127}-2e^{237}-e^{156}+e^{246},$<br>$e^{256},e^{127}-e^{237}-e^{156}+e^{345},-e^{247}-e^{157}+e^{346}$ | | |
| $0,0,12,13,23,15+24,16+26+34-35$ | $e_7$ | $[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{145},e^{146},e^{234},e^{235},e^{127}+e^{136}+e^{236},$<br>$e^{137}+e^{237},e^{127}+e^{136}+e^{245},e^{246}+2e^{137}-e^{156},$<br>$e^{256},e^{345}+e^{137}-e^{156},-e^{346}+e^{356}+e^{247}+e^{157}$ | | |
| $0,0,0,12,14+23,15-34,16-35$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},-e^{34}+e^{15}]$ |
| $e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{126}+e^{145},e^{146},e^{234},e^{126}+e^{235},e^{127}+e^{236},$<br>$e^{237}+e^{147}+e^{156},e^{245},-e^{127}+e^{345},-2e^{147}+e^{346}-e^{156},-e^{347}+e^{356}+e^{157}$ | | |
| $0,0,0,12,14+23,15-34,16+23-35$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},e^{15}-e^{34}]$ |
| $e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{126},e^{146},e^{234},e^{235}+e^{126},$<br>$e^{236}+e^{127},e^{237}+e^{126}+e^{156}+e^{147},e^{245},e^{345}-e^{127},$<br>$e^{346}-2e^{126}-e^{156}-2e^{147},e^{356}-e^{347}+e^{127}+e^{157}$ | | |
| $0,0,0,12,14+23,15-34,16+24-35$ | $e_7$ | $[0,0,0,e^{12},e^{14}+e^{23},e^{15}-e^{34}]$ |
| $e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},-e^{126}+e^{137},e^{126}+e^{145},e^{146},$<br>$e^{234},e^{126}+e^{235},e^{236}+e^{127},e^{156}+e^{147}+e^{237},e^{245},$<br>$e^{345}-e^{127},-e^{156}+e^{346}-2e^{147},-e^{246}+e^{356}-e^{347}+e^{157}$ | | |
| $0,0,12,13,23,24+15,16+14-25+34$ | $e_7$ | $[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$ |
| $e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},-e^{127}+e^{145},e^{146},e^{234},$<br>$e^{235},e^{236}-e^{137},-e^{127}+e^{237}-e^{147}+e^{156},-e^{137}+e^{245},-2e^{147}+e^{156}+e^{246},$<br>$e^{256},-e^{147}+e^{345},-e^{247}-e^{157}+e^{346}$ | | |
| $0,0,12,13,23,-14-25,16-35$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{23},-e^{14}-e^{25}]$ |
| $0,0,12,13,23,-14-25,16-35+25$ | $e_7(P)$ | $[0,0,e^{12},e^{13},e^{23},-e^{25}-e^{14}]$ |
| $0,0,0,12,14+23,15-34,16-23-35$ | $e_7$ | $[0,0,0,e^{12},e^{23}+e^{14},e^{15}-e^{34}]$ |
| $e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{126},e^{146},e^{234},e^{235}+e^{126},$<br>$e^{236}+e^{127},e^{237}-e^{126}+e^{156}+e^{147},e^{245},$<br>$e^{345}-e^{127},e^{346}+2e^{126}-e^{156}-2e^{147},e^{356}-e^{347}-e^{127}+e^{157}$ | | |

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